ISRAEL JOURNAL OF MATHEMATICS 99 (1997), 207–219

UNIVERSAL NON-COMPLETELY-CONTINUOUS OPERATORS

BY

MARIA GIRARDI*

Department of Mathematics, University of South Carolina
Columbia, SC 29208, USA
e-mail: girardi@math.scarolina.edu

AND

WILLIAM B. JOHNSON**

Department of Mathematics, Texas A&M University
College Station, TX 77843, USA
e-mail: johnson@math.tamu.edu

ABSTRACT

A bounded linear operator between Banach spaces is called **completely continuous** if it carries weakly convergent sequences into norm convergent sequences. Isolated is a universal operator for the class of non-completely-continuous operators from L_1 into an arbitrary Banach space, namely, the operator from L_1 into ℓ_∞ defined by

$$T_0(f) = \left(\int r_n f \, d\mu\right)_{n\geq 0},$$

where r_n is the nth Rademacher function. It is also shown that there does not exist a universal operator for the class of non-completely-continuous operators between two arbitrary Banach spaces. The proof uses the factorization theorem for weakly compact operators and a Tsirelson-like space.

Suppose that $\mathfrak{C}$ is a class of (always bounded, linear, between Banach spaces) operators so that an operator S is in $\mathfrak{C}$ whenever the domain of S is the domain

* Supported in part by NSF grant DMS-9306460. Participant, NSF Workshop in Linear Analysis & Probability, Texas A&M University (supported in part by NSF grant DMS-9311902).

** Supported in part by NSF grant DMS-9003550.

Received May 14, 1995

of some operator in $\mathfrak{C}$ and there exist operators A, B so that BSA is in $\mathfrak{C}$; the natural examples of such classes are all the operators that do not belong to a given operator ideal. A subset $\mathfrak{S}$ of such a class $\mathfrak{C}$ is said to be **universal** for $\mathfrak{C}$ provided for each U in $\mathfrak{C}$, some member of $\mathfrak{S}$ factors through U; that is, there exist operators A and B so that BUA is in $\mathfrak{S}$. In case $\mathfrak{S}$ is singleton, say $\mathfrak{S} = \{S\}$, we say that S is universal for $\mathfrak{C}$.

In order to study a class $\mathfrak{C}$ of operators, it is natural to try to find a universal subclass of $\mathfrak{C}$ consisting of specific, simple operators. For certain classes, such a subclass is known to exist. For example, Lindenstrauss and Pełczyński, who introduced the concept of universal operator, proved [LP] that the "summing operator" from ℓ_1 to ℓ_∞, defined by $\{a_n\}_{n=1}^{\infty} \mapsto \{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$, is universal for the class of non-weakly-compact operators, while in [J] it was pointed out that the formal identity from ℓ_1 to ℓ_∞ is universal for the class of non-compact operators.

An operator between Banach spaces is called **completely continuous** if it carries weakly convergent sequences into norm convergent sequences. The operator from L_1 into ℓ_∞ given by

$$T_0(f) = \left\{ \int r_n f \, d\mu \right\}_{n=0}^{\infty},$$

where r_n is the nth Rademacher function, is not completely continuous. We prove in Corollary 4 that T_0 is universal for the class of non-completely-continuous operators from an L_1-space; however, in Theorem 5 we show that there does not exist a universal non-completely-continuous operator.

Throughout this paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space of $\mathfrak{X}$, and $S(\mathfrak{X})$ the unit sphere of $\mathfrak{X}$. The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0,1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU] or [LT].

To crystalize the ideas in Theorem 1, we introduce some terminology. A system $\mathcal{A} = \{A_k^n \in \Sigma \colon n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots, 2^n\}$ is a **dyadic splitting** of $A_1^0 \in \Sigma^+$ if each A_k^n is partitioned into the two sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure for each admissible n and k. Thus the collection $\pi_n = \{A_k^n \colon k = 1, \ldots, 2^n\}$ of sets along the nth-level partition A_1^0 with π_{n+1} refining π_n and $\mu(A_k^n) = 2^{-n}\mu(A_1^0)$. To a dyadic splitting corresponds a (normalized) Haar system $\{h_j\}_{j\geq 1}$ along with its natural blocking $\{H_n\}_{n\geq 0}$ where

$$h_1 = \tfrac{1}{\mu(A_1^0)} 1_{A_1^0} \quad \text{and} \quad h_{2^n+k} = \tfrac{2^n}{\mu(A_1^0)} (1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}})$$

for $n = 0, 1, 2, \ldots$, $k = 1, \ldots, 2^n$, and $H_n = \{h_j\colon 2^{n-1} < j \leq 2^n\}$. The usual Haar system $\{\tilde{h}_j\}$ corresponds to the usual dyadic splitting $\{[\frac{k-1}{2^n}, \frac{k}{2^n})\}_{n,k}$. Let $L_1(\mathcal{A})$ be the closed subspace of L_1 with basis $\{h_j\}_{j\geq 1}$.

A set N in the unit sphere of the dual of a Banach space $\mathfrak{X}$ is said to norm a subspace $\mathfrak{X}_0$ within $\tau > 1$ if for each $x \in \mathfrak{X}_0$ there is $x^* \in N$ such that $\|x\| \leq \tau x^*(x)$. It is well known and easy to see that a sequence $\{\mathfrak{X}_j\}_{j\geqslant 1}$ of subspaces of $\mathfrak{X}$ forms a finite dimensional decomposition with constant at most τ provided that for each $n \in \mathbb{N}$ the space generated by $\{\mathfrak{X}_1, \ldots, \mathfrak{X}_n\}$ can be normed by a set from $S(\mathfrak{X}_{n+1}^{\perp})$ within $\tau_n > 1$ where $\Pi\tau_n \leq \tau$.

To help demystify Theorem 1, we examine more closely the operator $T_0\colon L_1 \to \ell_\infty$ given above. This operator does more than just map the Rademacher functions $\{r_n\}$ to the standard unit vectors $\{e_n\}$ in ℓ_∞ (which suffices to guarantee that it is not completely continuous). Let x_n^* be the nth unit vector of ℓ_1, viewed as an element in the dual of ℓ_∞. For the usual dyadic splitting of the unit interval, r_n is just the sum of the Haar functions in H_n, properly normalized. Thus $1 = \|T_0 r_n\| = x_n^*(T_0 r_n)$ follows from the stronger condition that

$$x_n^*(T_0 h) = \delta_{n,m} \qquad \text{for each } h \in H_m.$$

Note that $T_0^* x_n^*$ is just r_n, which as a sequence in L_1^* is weak*-null and equivalent to the unit vector basis of ℓ_1. Since T_0 maps each element in H_n to e_n, the collection $\{\text{sp } T_0 H_n\}$ forms a finite dimensional decomposition. Theorem 1 states that each non-completely-continuous operator T on L_1 behaves like the operator T_0 in the sense that there is some dyadic splitting of some subset of $[0, 1]$ so that the corresponding Haar system with T enjoys the above properties of the usual Haar system with T_0.

THEOREM 1: *Let Y be a subset of $S(\mathfrak{X}^*)$ that norms $\mathfrak{X}$ within some fixed constant greater than one and let $\mathcal{Y}$ be a subspace of $\mathfrak{X}^*$ that contains Y. If the operator $T\colon L_1 \to \mathfrak{X}$ is not completely continuous and $\{\tau_n\}_{n\geq 0}$ is a sequence of numbers larger than 1, then there exist*

(A) *a dyadic splitting $\mathcal{A} = \{A_k^n\}$,*

(B) *a sequence $\{x_n^*\}_{n\geq 0}$ in $S(\mathfrak{X}^*) \cap \mathcal{Y}$,*

(C) *a finite set $\{z_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ for each $n \geq 0$,*

such that for the Haar system $\{h_j\}_{j\geq 1}$ and the blocking $\{H_n\}_{n\geq 0}$ corresponding to $\mathcal{A}$, for some $\delta > 0$, and each $n, m \geq 0$,

(1) *$x_n^*(Th) = \delta \cdot \delta_{n,m}$ for each $h \in H_m$,*

(2) $\{T^*x_n^*\}$ *is weak*-null in* L_∞,

(3) $\{T^*x_n^*\}$ *is equivalent to the unit vector basis of* ℓ_1,

(4) $\{z_{n,i}^*\}_{i=1}^{p_n}$ *norms* $\mathrm{sp}(\bigcup_{j=0}^n TH_j)$ *within* τ_n,

(5) $TH_{n+1} \subset^\perp \{z_{n,i}^*\}_{i=1}^{p_n}$.

Note that condition (3) *implies that* $\{x_n^*\}$ *is also equivalent to the standard unit vector basis of* ℓ_1. *If* $\Pi\tau_n$ *is finite, then the last two conditions guarantee that* $\{\mathrm{sp}\, TH_n\}_{n\geq 0}$ *forms a finite dimensional decomposition with constant at most* $\Pi\tau_n$.

The proof uses the following two standard lemmas.

LEMMA 2: *Let* $E = \mathrm{sp}\{x_i\}_{i=0}^m$ *be a finite dimensional subspace of a Banach space* $\mathfrak{X}$ *and let* $\mathcal{Y}$ *be a total subspace of* $\mathfrak{X}^*$. *For each* $\epsilon > 0$ *there exists* $\eta > 0$ *such that if* $y^* \in \mathfrak{X}^*$ *satisfies* $|y^*(x_i)| < \eta$ *for each* $1 \leq i \leq m$, *then there exists* $x^* \in E^\perp$ *of norm* 0 *or* $\|y^*\|$ *such that* $\|x^* - y^*\| < \epsilon$. *Furthermore, if* y^* *is in* $\mathcal{Y}$ *then* x^* *can be taken to be in* $\mathcal{Y}$.

Proof of Lemma 2: Assume, without loss of generality, $\{x_i\}_{i=0}^m$ is linearly independent. Consider the isomorphism $l\colon E \to \ell_1^m$ that takes x_i to the ith unit basis vector of ℓ_1^m and let P be a projection from $\mathfrak{X}$ onto E that is $w(\mathcal{Y})$-continuous, so that P^*E^* is a subspace of $\mathcal{Y}$. Such a projection exists because $\mathcal{Y}$ is total. Then $\tilde{x}^* \equiv y^* \cdot (I_{\mathfrak{X}} - P)$ is in $E^\perp$. It is easy to check that for $\eta = \frac{\epsilon}{3\,\|l\|\,\|P\|}$, if $|y^*(x_i)| < \eta$ for each i, then $\|\tilde{x}^* - y^*\| \leq \frac{\epsilon}{3}$. If $\|\tilde{x}^*\| = 0$, then let $x^* = \tilde{x}^*$. Otherwise, let $x^* = (\|y^*\| \,/\, \|\tilde{x}^*\|)\ \tilde{x}^*$. Then $\|x^* - y^*\| \leq 2\|\tilde{x}^* - y^*\|$. Thus x^* does what it should do. ∎

Recall that the extreme points of $B(L_\infty)$ are just the ± 1-valued measurable functions.

LEMMA 3: *If* $\{f_i\}_{i=0}^n$ *is a finite subset of* L_1, $\{\alpha_i\}_{i=0}^n$ *are scalars, and*

$$S = \left\{ g \in B(L_\infty)\colon \int f_i g\, d\mu = \alpha_i \ \text{ for each } \ 0 \leq i \leq n \right\},$$

then $\mathrm{ext}\, S = S \cap \mathrm{ext}\, B(L_\infty)$, *where* ext *denotes the extreme points of a set. Also, if* S *is non-empty then so is* $\mathrm{ext}\, S$.

Specifically, we use the following version of this extreme point argument lemma.

LEMMA 3′: *If $F = \{f_1, \dots, f_n\}$ and there exists g in $B(L_\infty) \cap F^\perp$ such that $\int f_0 g\, d\mu \geq \alpha_0 > 0$, then there exists a ± 1-valued function u in $B(L_\infty) \cap F^\perp$ such that $\int f_0 u\, d\mu = \alpha_0$.*

Proof of Lemma 3: Consider, if there is one, a function g in S for which there exists a subset A of positive measure and $\epsilon > 0$ such that $-1+\epsilon < g 1_A < 1-\epsilon$. Since the set $\{f \in L_\infty \colon |f| \leq 1_A\} \cap \{f_i\}_{i=0}^{n}{}^\perp$, is infinite dimensional, it contains a non-zero element h of norm less than ϵ. But then $g \pm h \in S$ and so g is not an extreme point of S. Thus ext $S = S \cap \text{ext } B(L_\infty)$.

Since S convex and weak*-compact in L_∞, if S is non-empty then so is ext S.

As for the last claim of the lemma, just note that if $g \in B(L_\infty) \cap F^\perp$ satisfies $\int f_0 g\, d\mu \equiv \beta \geq \alpha_0 > 0$, then $\frac{\alpha_0}{\beta}\, g$ is in the set S where $\alpha_i = 0$ for $i > 0$. By the first part of the lemma, any extreme point u of S will do. ∎

Although the proof of Theorem 1 is somewhat technical, the overall idea is simple. Since T is not completely continuous, we start by finding a weakly convergent sequence $\{g_n\}$ in L_1 and norm one functionals y_n^* such that $\delta_0 \leq y_n^*(T g_n)$. Each x_n^* will be a small perturbation of some $y_{j_n}^*$. Conditions (2) and (3) can be arranged by standard arguments.

Now the proof gets technical. We begin by finding a subset A_1^0 where the L_∞ function $(T^* y_n^*) g_n$, which in the motivating example of T_0 is the function $r_n r_n$, is large in some sense. We then proceed by induction on the level n. Given a finite dyadic splitting up to nth-level provides the subsets $\{H_m\}_{m=0}^{n}$ of corresponding Haar functions. We need to split each A_k^n into 2 sets A_{2k-1}^{n+1} and A_{2k}^{n+1} (thus finding h_{2^n+k}) and find the desired functionals so that all works. It is easy to find the functionals to satisfy condition (4). In the search for x_{n+1}^*, apply Lemma 2 to the set E given in (†) so that we need only to almost (within some η) satisfy (1-i′) for some y_j^*; for then we can perturb y_j^* to find x_{n+1}^* that satisfies (1-i′) exactly. Next, for each A_k^n, apply Lemma 3′ with F as given in (‡) and $f_0 = T^* y_j^* 1_{A_k^n}$ and g being a small perturbation of $g_j 1_{A_k^n}$. All is set up so that such a perturbation exists for a j (dependent on n but independent of k) sufficiently large enough. Now Lemma 3′ gives that desired ± 1-valued Haar-like function that yields the desired splitting of the $(n+1)$th-level. The sets F_k^n are chosen exactly so that conditions (1-ii′), (1-iii′), and (5′) hold.

Proof of Theorem 1: Let $T \colon L_1 \to \mathfrak{X}$ be a norm one operator that is not completely continuous. Then there is a sequence $\{g_n\}$ in L_1 and a sequence $\{y_n^*\}$

in $S(\mathfrak{X}^*) \cap \mathcal{Y}$ satisfying:

(a) $\|g_n\|_{L_\infty} \leq 1$,

(b) g_n is weakly null in L_1,

(c) $\delta_0 \leq y_n^* (T g_n)$ for some $\delta_0 > 0$.

Using (a), (b), and (c) along with Rosenthal's ℓ_1 theorem [cf. LT, Prop. 2.e.5], by passing to a further subsequence, we also have that

(d) $\{T^* y_n^*\}$ is equivalent to the standard unit vectors basis of ℓ_1.

Since $B(L_\infty)$ is weak* sequentially-compact in L_∞, by passing to a subsequence and considering differences we may assume that

(e) $T^* y_n^*$ is weak*-null in L_∞,

where (d) allows normalization of the new y_n^*'s so as to keep them in $S(X^*)$ and, used with care, (b) ensures that (c) still holds for some (new) positive δ_0. But $\{(T^* y_n^*) \cdot g_n\}$ is also in $B(L_\infty)$ and so, by passing to yet another subsequence, we have that

(f) $\{(T^* y_n^*) \cdot g_n\} \to h$ weak* in L_∞

for some $h \in L_\infty$.

Since $\int h\, d\mu \geq \delta_0$, the set $A \equiv [h \geq \delta_0]$ has positive measure. We may assume, by replacing y_n^* by $-y_n^*$ and g_n by $-g_n$ when needed, that $\|T^* y_n^* \mid_A \|_{L_\infty} =$ ess sup $T^* y_n^* \mid_A$. So from (a) and (f) it follows that $\delta_0 \leqslant \liminf \operatorname{ess\,sup} T^* y_n^* \mid_A$ while from (e) it follows that $\limsup \mu[T^* y_n^* \mid_A \geqslant \delta_0 - \eta] < \mu(A)$ for each $0 < \eta < \delta_0$. Thus, since the closure of the set

$$\left\{ \frac{\int_E f\, d\mu}{\mu(E)} : E \subset A, E \in \Sigma^+ \right\}$$

is the interval [ess inf f, ess sup f], there is a subset A_1^0 of A with positive measure and j_0 such that $y_{j_0}^* T(1_{A_1^0}) = \delta \mu(A_1^0)$ for some positive δ less than δ_0, say $\delta \equiv \delta_0 - 3\epsilon$. Put $x_0^* = y_{j_0}^*$ and $H_0 \equiv \{h_1\} = \left\{ 1_{A_1^0} / \mu(A_1^0) \right\}$.

We shall construct, by induction on the level n, a dyadic splitting of A_1^0 along with the desired functionals. Towards this, take a decreasing sequence $\{\epsilon_n\}_{n \geq 0}$ of positive numbers such that $\epsilon_0 < \epsilon$ and $\sum \epsilon_n < \delta_0 / 2K$ where K is the basis constant of $\{T^* y_n^*\}$. The sequence $\{x_n^*\}$ will be chosen such that $\|x_n^* - y_{j_n}^*\| \leq \epsilon_n$ for some increasing sequence $\{j_n\}_n$ of integers, which will ensure conditions (2) and (3). Note that condition (1) is equivalent to the following 3 conditions holding:

(1-i) $x_n^*(Th) = 0$ for $h \in H_m$ and $0 \leq m < n$,

(1-ii) $x_m^*(Th) = 0$ for $h \in H_n$ and $0 \le m < n$,
(1-iii) $x_n^*(Th) = \delta$ for $h \in H_n$,

for each n. Clearly these three conditions hold for $n = 0$. Fix $n \ge 0$.

Suppose that we are given a finite dyadic splitting $\{A_k^m : m = 0, \ldots, n$ and $k = 1, \ldots, 2^m\}$ of A_1^0 up to nth-level, which gives the subsets $\{H_m\}_{m=0}^n$ of corresponding Haar functions. Thus we can find a finite set $\{z_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ such that $\{z_{n,i}^*\}_{i=1}^{p_n}$ norms $\mathrm{sp}(\bigcup_{j=0}^n TH_j)$ within τ_n. Suppose that we are also given $\{x_m^*\}_{m=0}^n$ in $\mathcal{Y} \cap S(X^*)$ such that the three subconditions of (1) hold and, if $k = 1, 2, \ldots, n$, then $\|x_k^* - y_{j_k}^*\| \le \epsilon_k$ for some j_k.

We shall find x_{n+1}^* along with $j_{n+1} > j_n$ such that $\|x_{n+1}^* - y_{j_{n+1}}^*\| \le \epsilon_{n+1}$ and we shall partition, for each $1 \le k \le 2^n$, the set A_k^n into 2 sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure (thus finding h_{2^n+k} and so finding the corresponding set $\{H_{n+1}\}$) such that

(1-i′) $x_{n+1}^*(Th) = 0$ for $h \in H_m$ and $0 \le m < n+1$,
(1-ii′) $x_m^*(Th) = 0$ for $h \in H_{n+1}$ and $0 \le m < n+1$,
(1-iii′) $x_{n+1}^*(Th) = \delta$ for $h \in H_{n+1}$,
(5′) $TH_{n+1} \subset {}^{\perp}\{z_{n,i}^*\}_{i=1}^{p_n}$.

Towards this, apply Lemma 2 to

$$E \equiv \{Th\colon h \in H_m \ , \ 0 \le m \le n\} \tag{†}$$

and ϵ_{n+1} to find the corresponding η_{n+1}. Let

$$F_k^n = \{1_{A_k^n}\} \cup \{T^* x_m^* 1_{A_k^n}\}_{m=0}^n \cup \{T^* z_{n,i}^* 1_{A_k^n}\}_{i=1}^{p_n} \subset L_1 \tag{‡}$$

and $F_n = \mathrm{sp}\left[\ \bigcup_{k=1}^{2^n} F_k^n\ \right]$.

Pick $j \equiv j_{n+1} > j_n$ so large that for $k = 1, \ldots, 2^n$

(g) $\left|(T^* y_j^*)\, h\right| < \eta_{n+1}$ for all $h \in \bigcup_{m=0}^n H_m$,
(h) $\left|\int_\Omega g_j f \, d\mu\right| \le \frac{\epsilon}{3}\|f\|$ for all f in F_n,
(i) $\int_{A_k^n} T^* y_j^* \cdot g_j \, d\mu \ge (\delta_0 - \epsilon)\, \mu(A_k^n)$.

Condition (g) follows from (e), condition (h) follows from (b) and the fact that F_n is finite dimensional, condition (i) follows from (f) and the definition of A.

By Lemma 2 and (g), there is $x_{n+1}^* \in S(\mathfrak{X}^*) \cap \mathcal{Y}$ such that $\|x_{n+1}^* - y_{j_{n+1}}^*\|$ is at most ϵ_{n+1} and $x_{n+1}^* Th = 0$ for each $h \in \bigcup_{m=0}^n H_m$. Thus (1-i′) holds.

Condition (h) gives that the L_∞-distance from g_j to

$$F_n^{\perp} \equiv \{g \in L_\infty\colon \int_\Omega fg \, d\mu = 0 \text{ for each } f \in F_n\}$$

is at most $\epsilon/3$. So there is $\tilde{g}_j \in F_n^{\perp} \cap B(L_\infty)$ such that $\|\tilde{g}_j - g_j\|_{L_\infty}$ is less than ϵ. Clearly $\tilde{g}_j 1_{A_k^n} \in F_k^{n\,\perp} \cap B(L_\infty)$ for each admissible k. By condition (i), for each admissible k,

$$\int_\Omega \left(T^* x_{n+1}^*\right) \cdot \left(\tilde{g}_j 1_{A_k^n}\right)\, d\mu \geq \delta \mu(A_k^n)$$

and so, by Lemma 3, there exists a function $u_k^n \in B(L_\infty) \cap F_k^{n\,\perp}$ such that

$$(*) \qquad \int_\Omega \left(T^* x_{n+1}^*\right) \cdot \left(u_k^n\right)\, d\mu = \delta \mu(A_k^n)$$

and u_k^n is of the form $1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}}$ for 2 disjoint sets A_{2k-1}^{n+1} and A_{2k}^{n+1} whose union is A_k^n. Furthermore, A_{2k-1}^{n+1} and A_{2k}^{n+1} are of equal measure since $1_{A_k^n} \in F_k^n$. Since $u_k^n \in F_k^{n\,\perp}$, conditions (1-ii′) and (5′) hold. Condition (1-iii′) is just (*). ∎

Theorem 1 contains much information. For example, the next corollary crystallizes the role of the previously mentioned operator T_0.

COROLLARY 4: *If the operator $T\colon L_1 \to \mathfrak{X}$ is not completely continuous, then there exist an isometry A and an operator B such that the following diagram commutes:*

$$\begin{array}{ccc} L_1 & \xrightarrow{\ T\ } & \mathfrak{X} \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ L_1 & \xrightarrow{\ T_0\ } & \ell_\infty \end{array}$$

Furthermore, if $\mathfrak{X}$ is separable, then T_0 and B may be viewed as operators into c_0.

Proof of Corollary 4: Let j_1 be the natural injection of $L_1(\mathcal{A})$ into L_1, let $\mathfrak{X}_0$ be the norm closure of $T\left(j_1\, L_1\left(\mathcal{A}\right)\right)$, and let $\tilde{x}_n^*$ be the restriction of x_n^* to $\mathfrak{X}_0$.

Since $\{T^* x_n^*\}$ is weak*-null in L_∞, $\tilde{x}_n^*$ is weak*-null in $\mathfrak{X}_0^*$. Thus the mapping $U\colon \ell_1 \to \mathfrak{X}_0^*$ that take the nth unit basis vector of ℓ_1 to $\tilde{x}_n^*$ is weak* to weak* continuous and so U is the adjoint of the operator $S\colon \mathfrak{X}_0 \to c_0$ where $S(x) = \left(\tilde{x}_n^*\left(x\right)\right)_{n\geq 0}$.

Consider the (commutative) diagram:

$$\begin{array}{ccccc} L_1 & \xrightarrow{T} & \mathfrak{X} & & \\ {\scriptstyle j_1}\uparrow & & \uparrow{\scriptstyle j_2} & & \\ L_1(\mathcal{A}) & \xrightarrow{T_{\mathcal{A}}} & \mathfrak{X}_0 & & \\ {\scriptstyle R}\uparrow & & \downarrow{\scriptstyle S} & & \\ L_1 & \longrightarrow & c_0 & \xrightarrow{j_3} & \ell_\infty \end{array}$$

where R: $L_1 \to L_1(\mathcal{A})$ is the natural isometry that takes a usual Haar function $\widetilde{h}_j$ in L_1 to the corresponding associated Haar function h_j in $L_1(\mathcal{A})$, the maps j_i are the natural injections, and $T_{\mathcal{A}}$ is such that the upper square commutes.

For an arbitrary space $\mathfrak{X}$, since ℓ_∞ is injective, the operator $j_3 S$ extends to an operator $\tilde{S}$: $\mathfrak{X} \to \ell_\infty$. For a separable space $\mathfrak{X}$, since c_0 is separably injective, this extension $\tilde{S}$ may be viewed as taking values in c_0.

Let $A = j_1 R$ and $B = \frac{1}{\delta}\tilde{S}$. Then $BTA(\widetilde{h}_j) = \frac{1}{\delta}\left(\tilde{x}_n^*\left(Th_j\right)\right)_{n\geq 0}$. Property 1 of Theorem 1 gives that $BTA = T_0$. ∎

Corollary 4 says that, viewed as an operator into ℓ_∞ (respectively, into c_0), T_0 is universal for the class of non-completely-continuous operators from L_1 into an arbitrary (respectively, separable) Banach space.

THEOREM 5: *There does not exist a universal operator for the class of non-completely-continuous operator.*

The proof of the nonexistence of such an operator uses the existence of a factorization through a reflexive space for a weakly compact operator.

Proof: Suppose that there did exist a universal non-completely-continuous operator, say T_1: $\mathfrak{X} \to \mathcal{Z}$ where $\mathfrak{X}$ and $\mathcal{Z}$ are Banach spaces. Then there is a sequence $\{x_n\}$ in $\mathfrak{X}$ of norm one elements that converge weakly to zero but whose images $\{T_1 x_n\}$ are uniformly bounded away from zero. Furthermore, by passing to a subsequence, we also have that $\{T_1 x_n\}$ is a basic sequence in $\mathcal{Z}$.

The first step of the proof uses T_1 to construct a "nice" universal non-completely-continuous operator. By Corollary 7 in [DFJP], there exists a reflexive space $\mathcal{Y}$ with a normalized unconditional basis $\{y_n\}$ such that the map S: $\mathcal{Y} \to \mathfrak{X}$ that sends y_n to x_n is continuous. Consider the map U: $\mathcal{Z} \to \ell_\infty$ that sends z to $(z_n^*(z))$ where $\{z_n^*\}$ is a bounded sequence in $\mathcal{Z}^*$ such that $\{T_1 x_n, z_n^*\}$

is a biorthogonal system. The map $I_{\mathcal{Y}} \equiv UT_1S$ sends y_n to the nth unit vector of ℓ_∞. The reflexivity of $\mathcal{Y}$ guarantees that $I_{\mathcal{Y}}$ is not completely continuous. Since $I_{\mathcal{Y}}$ factors through the universal operator T_1, the operator $I_{\mathcal{Y}}$ must also be universal. We now work with this "nice" operator $I_{\mathcal{Y}}$.

For any linearly independent finite set $\{x_k\}_{k=1}^n$, let $\mathcal{D}\{x_k\}_{k=1}^n$ be the norm of the operator from the span of $\{x_k\}_{k=1}^n$ to ℓ_1^n that sends x_k to the kth unit vector of ℓ_1^n. Set $d_n = \mathcal{D}\{y_k\}_{k=1}^n$. Reflexivity of $\mathcal{Y}$ gives that d_n tends to infinity. Let T be a (reflexive) Tsirelson-like space with normalized unconditional basis $\{t_n\}$ such that for all finite subsets F of natural numbers,

$$\mathcal{D}\{t_n\}_{n\in F} \leq \max\left\{2, \sqrt{d_{|F|}}\right\},$$

where $| F |$ is the cardinality of F. For example, $\{t_n\}$ can just be an appropriately chosen subsequence of the usual basis of the usual Tsirelson space [cf. CS, Chapter I]. Consider the non-completely-continuous map $I_T: T \to \ell_\infty$ that sends t_n to the nth unit vector of ℓ_∞. By the universality of $I_{\mathcal{Y}}$, there exist maps A and B such that the following diagram commutes:

$$\begin{array}{ccc} T & \xrightarrow{I_T} & \ell_\infty \\ {\scriptstyle A}\uparrow & & \downarrow{\scriptstyle B} \\ \mathcal{Y} & \xrightarrow{I_{\mathcal{Y}}} & \ell_\infty \end{array}$$

Since each $I_{\mathcal{Y}}(y_n)$ is of norm one, there exists $\delta > 0$ such that $\delta < \|I_T A y_n\|$ for each n. Each Ay_n is of the form

$$Ay_n = \sum_{m=1}^{\infty} \alpha_{n,m}\, t_m$$

and so there is a sequence $\{m(n)\}_n$ of natural numbers such that $\delta <| \alpha_{n,m(n)} |$. Since $\{y_n\}$ tends weakly to zero, for each m the set of all n for which $m(n) = m$ is finite. Thus by replacing $\mathcal{Y}$ with the closed span of a suitable subsequence of $\{y_n\}$, we may assume that the $m(n)$'s are distinct.

Let T_* be the subspace of T spanned by $\{t_{m(n)}\}_n$. Since $\{y_n\}$ and $\{t_{m(n)}\}$ are both unconditional bases, by the diagonalization principle [cf. LT, Prop. 1.c.8], the correspondence $y_n \mapsto \alpha_{n,m(n)}t_{m(n)}$ extends to an operator D: $\mathcal{Y} \to T_*$. Since $\{t_{m(n)}\}$ is an unconditional basis and $\delta <| \alpha_{n,m(n)} |$, the correspondence $\alpha_{n,m(n)}\, t_{m(n)} \mapsto t_{m(n)}$ extends to an operator $M: T_* \to T_*$.

By the definition of d_n, there exists a sequence $\{\beta_i^n\}_{i=1}^n$ such that $\sum_{i=1}^n |\beta_i^n| = 1$ and

$$\|\sum_{i=1}^n \beta_i^n y_i\|_{\mathcal{Y}} = \frac{1}{d_n}.$$

By the choice of T, for large n,

$$\frac{1}{\sqrt{d_n}} \le \|\sum_{i=1}^n \beta_i^n \, t_{m(i)}\|_{T_*}.$$

Since $MD: \mathcal{Y} \to T_*$ maps y_n to $t_{m(n)}$,

$$\|\sum_{i=1}^n \beta_i^n t_{m(i)}\|_{T_*} \le \|MD\| \, \|\sum_{i=1}^n \beta_i^n y_i\|_{\mathcal{Y}}.$$

This gives that

$$\frac{1}{\sqrt{d_n}} \le \frac{\|MD\|}{d_n},$$

which cannot be since d_n tends to infinity. ∎

The first two paragraphs of the proof of Theorem 5 yield part (a) of the next proposition. Part (b) follows from similar considerations and the Gurarii–James theorem [Ja, Thm. 2].

PROPOSITION 6:

(a) *Let $\mathfrak{S}$ be the collection of all formal identity operators into ℓ_∞ from reflexive sequence spaces for which the unit vectors form a normalized unconditional basis. Then $\mathfrak{S}$ is universal for the class of all non-completely-continuous operators.*

(b) *The collection $\{I: \ell_p \to \ell_\infty; 1 < p < \infty\}$ of formal identity operators is universal for the class of all non-completely-continuous operators whose domain is superreflexive.*

Recall that a Banach space $\mathfrak{X}$ has the Radon–Nikodým Property (RNP) [respectively, is strongly regular, has the Complete Continuity Property (CCP)] if each bounded linear operator from L_1 into $\mathfrak{X}$ is representable [respectively, strongly regular, completely continuous]. The books [DU], [GGMS], and [T] contain splendid surveys of these properties. Here we only recall that a representable operator is strongly regular and a strongly regular operator is completely continuous. The first paragraph of the proof of Theorem 1 uses elementary methods to construct, from an operator $T\colon L_1 \to \mathfrak{X}$ that is not completely continuous, a copy of ℓ_1 in the closed span of a norming set of $\mathfrak{X}$. On a much deeper level, the following fact is well-known.

FACT: *The following are equivalent.*

(1) *ℓ_1 embeds into $\mathfrak{X}$.*
(2) *L_1 embeds into $\mathfrak{X}^*$.*
(3) *$\mathfrak{X}^*$ fails the CCP.*
(4) *$\mathfrak{X}^*$ is not strongly regular.*

The well-known equivalence of (1) and (2) was shown by Pełczyński [P, for separable $\mathfrak{X}$] and Hagler [H, for non-separable $\mathfrak{X}$]. The other downward implications follow from the definitions. Bourgain [B] used a non-strongly-regular operator into a dual space to construct a copy of ℓ_1 in the pre-dual. Here the authors wish to formalize the following essentially known fact which, to the best of our knowledge, has not appeared in print as such.

FACT: *The following are equivalent.*

(1) *$\mathfrak{X}$ has trivial type.*
(2) *$\mathfrak{X}$ fails super CCP.*
(3) *$\mathfrak{X}$ is not super strongly regular.*

Proof: To see that (1) implies (2), recall that $\mathfrak{X}$ has trivial type if and only if ℓ_1 is finitely representable in $\mathfrak{X}$ and that L_1 is finitely representable in ℓ_1. Thus, if $\mathfrak{X}$ has trivial type, then L_1 is finitely representable in $\mathfrak{X}$ and so $\mathfrak{X}$ cannot have the super CCP. Property (3) formally follows from (2). Towards seeing that (3) implies (1), consider a space $\mathfrak{X}$ that is not strongly regular. From the above fact it follows that ℓ_1 embeds into $\mathfrak{X}^*$. Thus $\mathfrak{X}^*$ has trivial type, which implies the same for $\mathfrak{X}$. ∎

References

[B] J. Bourgain, *On martingales in conjugate Banach spaces*, unpublished.

[CS] Peter G. Casazza and Thaddeus J. Shura, *Tsirelson's Space*, Lecture Notes in Mathematics **1363**, Springer-Verlag, Berlin, 1989.

[DFJP] W. J. Davis, T. Figiel, W. B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, Journal of Functional Analysis **17** (1974), 311–327.

[D] Joseph Diestel, *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, Vol. 92, Springer-Verlag, New York–Berlin, 1984.

[DU] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Mathematics Surveys, no. 15, American Mathematical Society, Providence, R.I., 1977.

[GGMS] N. Ghoussoub, G. Godefroy, B. Maurey and W. Schachermayer, *Some topological and geometrical structures in Banach spaces*, Memoirs of the American Mathematical Society, no. 378, American Mathematical Society, Providence, R.I., 1987.

[H] James Hagler, *Some more Banach spaces which contain ℓ^1*, Studia Mathematica **46** (1973), 35–42.

[Ja] R. C. James, *Super-reflexive spaces with bases*, Pacific Journal of Mathematics **41** (1972), 409–419.

[J] W. B. Johnson, *A universal non-compact operator*, Colloquium Mathematicum **23** (1971), 267–268.

[LP] J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in $\mathcal{L}_p$-spaces and their applications*, Studia Mathematica **29** (1968), 275–326.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, New York–Berlin, 1977.

[P] A. Pełczyński, *On Banach spaces containing $L_1(\mu)$*, Studia Mathematica **30** (1968), 231–246.

[RS] Lawrence H. Riddle and Elias Saab, *On functions that are universally Pettis integrable*, Israel Journal of Mathematics **29** (1985), 509–531.

[T] Michel Talagrand, *Pettis integral and measure theory*, Memoirs of the American Mathematical Society, no. 307, American Mathematical Society, Providence, R.I., 1984.